Find two linearly independent series solutions of $(2x+1)y'' + xy' - 3y = 0$ about $x = 0$. **SCORE: _____ / 14 PTS**

ORDINARY POINT

$$(2x+1)\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n - 3\sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=0}^{\infty}2(n+2)(n+1)a_{n+2}x^{n+1} + \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n+1} - \sum_{n=0}^{\infty}3a_n x^n = 0$$

ALL ITEMS WORTH (1) POINT UNLESS OTHERWISE NOTED

$$\sum_{n=1}^{\infty}2(n+1)n a_{n+1}x^n + \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty}n a_n x^n - \sum_{n=0}^{\infty}3a_n x^n = 0$$

$$2(1)a_2 - 3a_0 + \sum_{n=1}^{\infty}\left[2(n+1)n a_{n+1} + (n+2)(n+1)a_{n+2} + na_n - 3a_n\right]x^n = 0$$

$$2a_2 - 3a_0 = 0 \qquad 2(n+1)n a_{n+1} + (n+2)(n+1)a_{n+2} + (n-3)a_n = 0,\ n \in \mathbb{Z}^+$$

$$a_2 = \frac{3}{2}a_0 \qquad \boxed{a_{n+2} = -\frac{2n(n+1)a_{n+1} + (n-3)a_n}{(n+2)(n+1)}},\ n \in \mathbb{Z}^+$$

$a_0 = 1,\ a_1 = 0$

$a_2 = \frac{3}{2}$

$$a_3 = -\frac{2(1)(2)(\frac{3}{2}) + (-2)0}{3(2)} = -1$$

$$a_4 = -\frac{2(2)(3)(-1) + (-1)(\frac{3}{2})}{4(3)} = \frac{\frac{27}{2}}{12} = \frac{9}{8}$$

$$y_1 = \underbrace{1}_{\frac{1}{2}} + \underbrace{\frac{3}{2}x^2}_{\frac{1}{2}} \underbrace{- x^3}_{\frac{1}{2}} + \underbrace{\frac{9}{8}x^4}_{\frac{1}{2}} + \cdots$$

$a_0 = 0,\ a_1 = 1$

$a_2 = 0$

$$a_3 = -\frac{2(1)(2)0 + (-2)1}{3(2)} = \frac{1}{3}$$

$$a_4 = -\frac{2(2)(3)(\frac{1}{3}) + (-1)0}{4(3)} = -\frac{1}{3}$$

$$a_5 = -\frac{2(3)(4)(-\frac{1}{3}) + 0(\frac{1}{3})}{5(4)} = \frac{2}{5}$$

$$y_2 = \underbrace{x}_{\frac{1}{2}} + \underbrace{\frac{1}{3}x^3}_{\frac{1}{2}} \underbrace{- \frac{1}{3}x^4}_{\frac{1}{2}} + \underbrace{\frac{2}{5}x^5}_{\frac{1}{2}}$$

Find two linearly independent series solutions of $2x^2y'' + (x^2 - x)y' - 2y = 0$ about $x = 0$. SCORE: _____ / 16 PTS

$$y'' + \left(\tfrac{1}{2} - \tfrac{1}{2x}\right)y' - \tfrac{1}{x^2}y = 0$$ SINGULAR POINT

$$x\left(\tfrac{1}{2} - \tfrac{1}{2x}\right) = \tfrac{1}{2}x - \tfrac{1}{2} \longrightarrow x = 0 \text{ IS RSP}$$

$$x^2\left(-\tfrac{1}{x^2}\right) = -1$$

$$\lim_{x\to 0}\left(\tfrac{1}{2}x - \tfrac{1}{2}\right) = -\tfrac{1}{2}, \quad \lim_{x\to 0}(-1) = -1$$

$$r(r-1) - \tfrac{1}{2}r - 1 = 0 \longrightarrow 2r(r-1) - r - 2 = 0$$

$$2r^2 - 3r - 2 = 0$$

$$(2r+1)(r-2) = 0$$

$$r = -\tfrac{1}{2}, 2$$

ALL ITEMS WORTH 1 POINT UNLESS OTHERWISE NOTED

$$2x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r-2} + (x^2 - x)\sum_{n=0}^{\infty}(n+r)a_nx^{n+r-1} - 2\sum_{n=0}^{\infty}a_nx^{n+r} = 0$$

$$\sum_{n=0}^{\infty}2(n+r)(n+r-1)a_nx^{n+r} + \sum_{n=0}^{\infty}(n+r)a_nx^{n+r+1} - \sum_{n=0}^{\infty}(n+r)a_nx^{n+r} - \sum_{n=0}^{\infty}2a_nx^{n+r} = 0$$

$$\sum_{n=1}^{\infty}(n+r-1)a_{n-1}x^{n+r}$$

$$2r(r-1)a_0x^r - ra_0x^r - 2a_0x^r + \sum_{n=1}^{\infty}\left[2(n+r)(n+r-1)a_n + (n+r-1)a_{n-1} - (n+r)a_n - 2a_n\right]x^{n+r} = 0$$

$$[2r(r-1) - r - 2]a_0x^r = 0$$

SAME AS AT TOP $\rightarrow r = -\frac{1}{2}, 2$

$$[2(n+r)(n+r-1) - (n+r) - 2]a_n + (n+r-1)a_{n-1} = 0, \quad n \in \mathbb{Z}^+$$

$$a_n = -\frac{(n+r-1)}{2(n+r)(n+r-1) - (n+r) - 2}a_{n-1}, \quad n \in \mathbb{Z}^+$$

$$r = 2: \ a_n = -\frac{n+1}{2(n+2)(n+1) - (n+2) - 2}a_{n-1} = -\frac{n+1}{n(2n+5)}a_{n-1}, \quad n \in \mathbb{Z}^+$$

$$a_0 = 1$$

$$a_1 = -\frac{2}{1\cdot 7}a_0 = -\frac{2}{1\cdot 7}$$

$$a_2 = -\frac{3}{2\cdot 9}a_1 = \frac{3\cdot 2}{2\cdot 1\cdot 9\cdot 7}$$

$$a_3 = -\frac{4\cdot 3\cdot 2}{3\cdot 2\cdot 1\cdot 11\cdot 9\cdot 7}$$

$$y_1 = x^2\left(1 - \frac{2}{1\cdot 7}x + \frac{3\cdot 2}{2\cdot 1\cdot 9\cdot 7}x^2 - \frac{4\cdot 3\cdot 2}{3\cdot 2\cdot 1\cdot 11\cdot 9\cdot 7}x^3\right)$$

$\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$

$$r = -\frac{1}{2}: \quad a_n = -\frac{n-\frac{3}{2}}{2(n-\frac{1}{2})(n-\frac{3}{2}) - (n-\frac{1}{2}) - 2}\, a_{n-1} = -\frac{2n-3}{2n(2n-5)}\, a_{n-1}, \quad n \in \mathbb{Z}^+$$

$$a_0 = 1$$

$$a_1 = -\frac{-1}{2(-3)}\, a_0 = -\frac{(-1)}{2(-3)}$$

$$a_2 = -\frac{1}{4(-1)}\, a_1 = -\frac{1(-1)}{4\cdot 2\cdot(-1)\cdot(-3)}$$

$$a_3 = -\frac{3}{6(1)}\, a_2 = -\frac{3(1)(-1)}{6\cdot 4\cdot 2\cdot 1\cdot(-1)\cdot(-3)}$$

$$y_2 = x^{-\frac{1}{2}}\left(\underbrace{1}_{\frac{1}{2}} - \underbrace{\frac{1}{2\cdot 3}x}_{\frac{1}{2}} + \underbrace{\frac{1}{4\cdot 2\cdot 3}x^2}_{\frac{1}{2}} + \underbrace{\frac{3}{6\cdot 4\cdot 2\cdot 3}x^3}_{\frac{1}{2}} + \cdots\right)$$